Lax Matrix

Sunday, January 30, 2022 11:18 AM

- Let V be a vector space. Then any linear operator on $\mathbb{C}^2 \otimes V$ can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D : V \to V$$

as $\mathbb{C}^2 \otimes V = \underbrace{\left[\begin{pmatrix}1\\0\end{pmatrix} \otimes V\right]}_{``V^1"} \oplus \underbrace{\left[\begin{pmatrix}0\\1\end{pmatrix} \otimes V\right]}_{``V^2"}$

$A : V^1 \to V^1, \quad B : V^1 \to V^2$

$C : V^2 \to V^1, \quad D : V^2 \to V^2$

Def Given $u \in \mathbb{C}$ (spectral parameter), let

$$L_{an}(u) = \begin{pmatrix} uI_2 + iS_n^z & iS_n^- \\ iS_n^+ & uI_2 - iS_n^z \end{pmatrix} : \mathbb{C}_a^2 \otimes V_n \to \mathbb{C}_a^2 \otimes V_n$$

Lax matrix for XXX spin chain

where $V_n = \mathbb{C}^2$ is the n-th site in

$$V = V_1 \otimes \cdots \otimes V_n \otimes \cdots \otimes V_L$$

$S_n^+, S_n^-, S_n^z : V_n \to V_n$ are the spin operators

$\boxed{\text{Prop}}$: In $\mathbb{C}_a^2 \otimes \mathbb{C}_b^2 \otimes V_n$, let

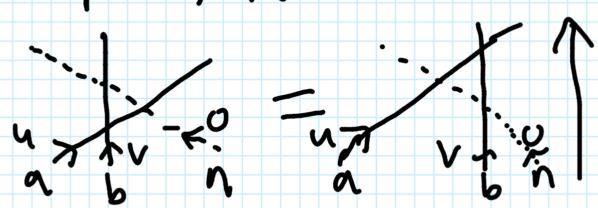
$\tilde{L}_{an}(u) = L_{an}(u) \otimes I_b$ ($\underbrace{\otimes \; \otimes}_{\text{acts by } L_{an}(u)}$; acts by I_2)

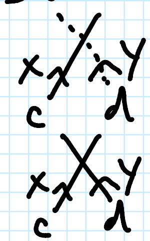
$\tilde{L}_{bn}(u) = I_a \otimes L_{bn}(u)$ ($\otimes \; \underbrace{\otimes}_{\text{acts by } L_{bn}(u)}$; acts by I_2)

Then have RLL relation

$$R_{ab}(u-v)\tilde{L}_{an}(u)\tilde{L}_{bn}(v) = \tilde{L}_{bn}(v)\tilde{L}_{an}(u)R_{ab}(u-v)$$

Graphically RLL relation is (read bot to top)





$= L_{cd}(x-y)$

$= R_{cd}(x-y)$

Monodromy Matrix

Sunday, January 30, 2022 11:18 AM

where $\underbrace{R_{ab}}_{\text{act on } a,b \text{ component}}(w) = \begin{pmatrix} w+i & 0 & 0 & 0 \\ 0 & w & i & 0 \\ 0 & i & w & 0 \\ 0 & 0 & 0 & w+i \end{pmatrix}_{ab}$

Warning: Frequently, $L_{ab}(u) = \widetilde{L_{ab}(u)}$ in books. In gen, a,b det comp of action, rest I_2.

Similarly, $S_n^+ = I \otimes \cdots \otimes S_n^+ \otimes \cdots \otimes I$

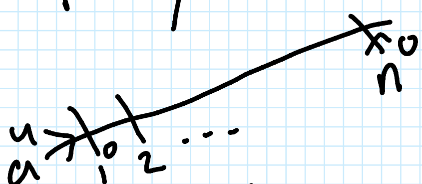
Def The monodromy matrix is

$$M_a(u) = L_{a1}(u) L_{a2}(u) \cdots L_{aL}(u)$$

is an op $\mathbb{C}^2_a \otimes \underbrace{V_1 \otimes \cdots \otimes V_L}_{\mathcal{V}} \to \mathbb{C}^2_a \otimes \underbrace{V_1 \otimes \cdots \otimes V_L}_{\mathcal{V}}$

$\Downarrow$

Graphically



$$M_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$A(u)$, etc: $V_1 \otimes \cdots \otimes V_L \to V_1 \otimes \cdots \otimes V_L$

Ex: $L = 2$, $M_a(u) = L_{a1}(u) L_{a2}(u)$

$$= \begin{pmatrix} u+iS_1^z & iS_1^- \\ iS_1^+ & u-iS_1^z \end{pmatrix} \begin{pmatrix} u+iS_2^z & iS_2^- \\ iS_2^+ & u-iS_2^z \end{pmatrix}$$

$\Rightarrow A(u) = (u+iS_1^z)(u+iS_2^z) - S_1^- S_2^+$

$B(u) = i(u+iS_1^z)S_2^- + iS_1^-(u - iS_2^z)$

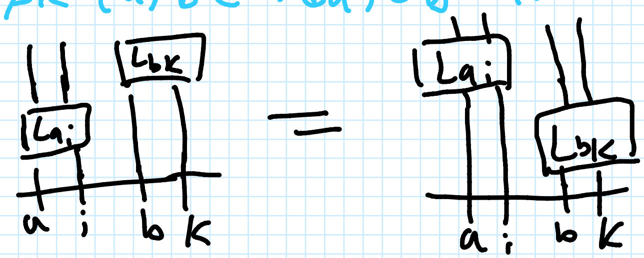
$C(u) = \dots$

$D(u) = \dots$

Key Insight: Actual form of $A(u)$, $B(u)$, etc doesn't matter, only need relations the operators satisfy for "Algebraic Bethe Ansatz"

Lemma 1: $L_{ai}(u) L_{bk}(v) = L_{bk}(v) L_{ai}(u)$ if $i \neq k$ ($a, b \leftrightarrow \mathbb{C}^2_a, \mathbb{C}^2_b$ auxiliary)

Pf:



Monodromy Matrix 2

Sunday, January 30, 2022 11:18 AM

Prop 2 We have the following RMM rel In $\mathbb{C}^2_a \otimes \mathbb{C}^2_b \otimes V$

$$R_{ab}(u-v)\widetilde{M_a}(u)\widetilde{M_b}(v) = \widetilde{M_b}(v)\widetilde{M_a}(u)R_{ab}(u-v) \quad (*)$$

Pf: $LHS = R_{ab}(L_{a1}L_{a2}\cdots L_{aL})(L_{b1}L_{b2}\cdots L_{bL})$

$\overset{L1}{=} (R_{ab}L_{a1}L_{b1})(L_{a2}\cdots L_{aL})(L_{b2}\cdots L_{bL})$

$\overset{RLL}{=} (L_{b1}L_{a1})R_{ab}(L_{a2}\cdots L_{aL})(L_{b2}\cdots L_{bL})$

$\overset{repeat}{=} (L_{b1}L_{a1})(L_{b2}L_{a2})\cdots(L_{bL}L_{aL})R_{ab}$

$\overset{L1}{=} (L_{b1}L_{b2}\cdots L_{bL})(L_{a1}L_{a2}\cdots L_{aL})R_{ab} = RHS$

(pf still works after adding back spectral par)

Rem: Local commutativity (RLL) $\Rightarrow$ global commutativity (RMM)

Key Cor: Prop 2 gives relations b/t $A(u)$, ...

- Namely using the matrix forms

$$\widetilde{M}_{a(u)} = M_a(u) \otimes I_b, \quad \widetilde{M}_b(v) = I_a \otimes M_b(v)$$

$$\text{LHS of } (*) = \begin{pmatrix} u-v+i & & & \\ & u-v & i & \\ & i & u-v & \\ & & & u-v+i \end{pmatrix} \overset{A}{\begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}} \overset{B}{\begin{pmatrix} A(v) & B(v) & & \\ C(v) & D(v) & & \\ & & A(v) & B(v) \\ & & C(v) & D(v) \end{pmatrix}}$$

Q: What is 1,4 entry of above?

A: $= \sum R_{1x}A_{xy}B_{y4}$ - only $R_{11} \neq 0$ in R_{1x}

$= \sum R_{11}A_{1y}B_{y4}$ - only $B_{34}, B_{44} \neq 0$

$= R_{11}A_{13}B_{34} = (u-v+i)B(u)B(v)$ - only $A_{11}, A_{13} \neq 0$

Q: What is 1,4 entry of BAR?

A: $= \sum B_{1x}A_{xy}R_{y4}$ - only $R_{44} \neq 0$ in R_{y4}

$= \sum B_{1x}A_{x4}R_{44}$ - only $B_{11}, B_{12} \neq 0$

$= B_{12}A_{24}R_{44}$ - only $A_{24}, A_{44} \neq 0$

$= (u-v+i)B(v)B(u)$

Transfer Matrix

Sunday, January 30, 2022 11:18 AM

- Thus Prop 2 tells us

$$B(u)B(v) = B(v)B(u)$$

which isn't obvious from def of $B(u)$ even when $L=2$.

- Looking at other entries give other useful relations for next week

---

<u>Def</u> The transfer matrix of XXX spin chain

$$T(u) = Tr_a M_a(u) = A(u) + D(u)$$

<u>Prop 3</u>: $T(u)T(v) = T(v)T(u)$

<u>Goal:</u> Find $e\vec{v}$, ev of $T(u)$

- Recall that in stat mech, we wanted to find the $e\vec{v}$, ev of transfer matrix to compute the partition function

- In quantum mechanics, we want to find $e\vec{v}$, ev of transfer matrix to solve Schrödinger's equation

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

where $\hat{H}$ is the (quantum) Hamiltonian

- It turns out $H_{xxx} \sim \frac{d}{du} \log T(u) \Big|_{u=i/2}$

$\Rightarrow$ e.v of $H_{xxx} = \frac{d}{du} \log \lambda(u) \Big|_{u=i/2}$

where $\lambda(u)$ = e.v of $T(u)$

$\Rightarrow$ e.$\vec{v}$ of H_{xxx} = e.$\vec{v}$ of $T(u)$